ELECTROSTATICS : Chapter 1

<u>Honours: Semester 2</u>

Paper: PHS-A-CC-2-3-TH

by

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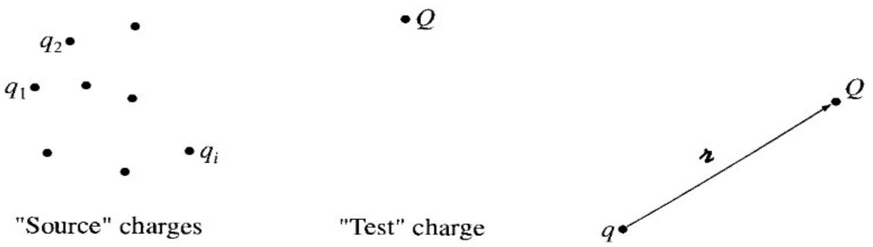
Notes: All Bold Fonts denote Vectors.

Theory of Superposition :

We have some electric charges, q_1, q_2, q_3, ...(source charges); what force do they exert on another charge, Q (test charge)?

The solution to this problem is facilitated by the principle of superposition, which states that the interaction between any two charges is completely unaffected by the presence of others. This means that to determine the force on Q, we can first compute the force $\mathbf{F_1}$, due to q_1 alone - ignoring all the others; then we compute the force $\mathbf{F_2}$, due to q_2 alone; and so on.

Finally, we take the vector sum of all these individual forces: $\mathbf{F} = \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} + ...$
Thus, if we can find the force on Q due to a single source charge q, we are, in principle, done.



Coulomb's Law :

The force on a test charge Q due to a single point charge q, which is at rest a distance r is given by the Coulombs law.

In general, the force of attraction (or repultion) between q and Q placed in free space is given by

$$\boxed{\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}.} \qquad \boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r'} \;;$$

$\boldsymbol{\mathscr{r}}$ is the separation vector from r (the location of q) to r' (the location of Q).

And ε_0 is the permitivity of free space. In SI units, where force is in Newtons (N), distance in meters (m), and charge in coulombs (C), $\varepsilon_0 = 8.85 \times 10^{-12}\ C^2N^{-1}M^{-2}$ (or, Farad/meter) .

Electrostatic field :

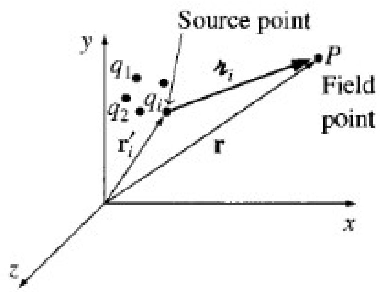
In electrostatiocs, we assume that all the source charges are stationary (though the test charge may be moving). The physics of stationary charges are known as Electrostatics.

Note : In electrostatics, isolated charge may be considered, but in magnetostatics, isolated magnetic poles are not possible.

If we have several point charges distributed over several distances as shown in the figure below, the total force on Q is

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{\mathscr{r}_1^2}\hat{\boldsymbol{\mathscr{r}}}_1 + \frac{q_2 Q}{\mathscr{r}_2^2}\hat{\boldsymbol{\mathscr{r}}}_2 + \ldots\right)$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1\hat{\boldsymbol{\mathscr{r}}}_1}{\mathscr{r}_1^2} + \frac{q_2\hat{\boldsymbol{\mathscr{r}}}_2}{\mathscr{r}_2^2} + \frac{q_3\hat{\boldsymbol{\mathscr{r}}}_3}{\mathscr{r}_3^2} + \ldots\right),$$



or

$$\boxed{\mathbf{F} = Q\mathbf{E},}$$

where

$$\mathbf{E}(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i.$$

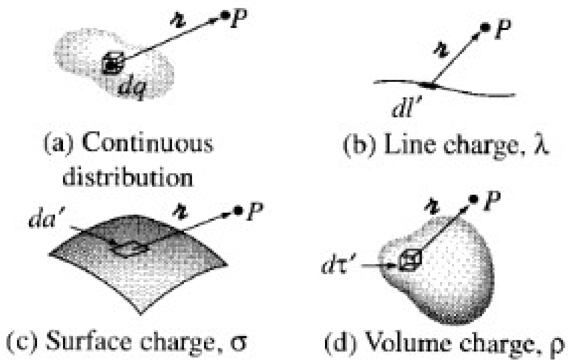
E is called the electric field of the source charges. Notice that it is a function of position (**r**) depend on the location of the field point P . But it makes no reference to the test charge Q. The electric field is a vector quantity that varies from point to point and is determined by the configuration of source charges. Physically, **E(r)** is the force per unit charge that would be exerted on a test charge, if you were to place one at P.

Therefore remember the following :

✔ The electric intensity (or field **E**) at any point P due to a point charge +q is defined as the force experienced by a unit +ve charge located at P due to a actual charge located at O.

✔ The electric potential at ant point P due to a point charge +q at A is defined as the work done in bringing a unit positive charge from infinity to the said point.

Electrostatic Field & Charge Density : Line, Surface and Volume charge density



The the electric field **E** assumes that the source of the field is a collection of discrete point charges q_i. If, instead, the charge is distributed continuously over some region, the sum becomes an integral (Fig. a) :

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{\mathscr{r}^2} \hat{\mathscr{r}}\, dq.$$

If the charge is spread out along a line with charge-per-unit-length λ (line charge density), then dq = λ dl' (where dl' is an element of length along the line (Fig. b); the electric field of a line charge is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{P}} \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2} \hat{\mathscr{r}}\, dl';$$

if the charge is smeared out over a surface (Fig. c), with charge-per-unit-area σ (surface charge density), then dq = σ da' (where da' is an element of area on the surface); for a surface charge,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{S}} \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2} \hat{\mathscr{r}}\, da';$$

and if the charge fills a volume (Fig. 2.5d), with charge-per-unit-volume ρ (volume charge density), then dq = ρ dτ' (where dτ' is an element of volume): the electric field of a volume charge is

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{\mathscr{r}^2} \hat{\mathscr{r}}\, d\tau'.}$$

Divergence of Fields :

1. Field Lines & Flux :

Let a point charge +q be located at any space-point. The lines of force will be emitted from +q and terminate at the ground. The lines of force may be extended upto infinity.

The lines of force contained per unit volume described about any point P is called the volume density of the lines of force ot field lines.

The lines of force travelling normally owtward from unit area is called the flus density of the lines of force. Thus, if **E** be the electric field intensity at an elementary surface 'ds', the normal outward flux from ds is : ***E.ds***

2. Gauss's Law & Theorem:

Statement : The total normal outward flux (or electrical induction) from any closed surface (S) containing a distribution od charges $\sum q$ is equal to the algebraic sum of all the charges included within S divided by the permittivity of free space ε_0.

Mathematically,

$$\boxed{\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}},}$$

where Q_{enc} is the total charge enclosed within the surface.

Proof :

Now let us assume, a point charge q is situated at the origin of a sphere, then the flux of E through a sphere of radius r is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \int \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{\mathbf{r}}\right) \cdot (r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}) = \frac{1}{\epsilon_0} q.$$

Now suppose that instead of a single charge at the origin, we have a bunch of charges scattered about. According to the principle of superposition, the total field is the (vector) sum of all the individual fields:

$$\mathbf{E} = \sum_{i=1}^{n} \mathbf{E}_i.$$

Therefore, the flux through a surface that encloses them all is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \sum_{i=1}^{n}\left(\oint \mathbf{E}_i \cdot d\mathbf{a}\right) = \sum_{i=1}^{n}\left(\frac{1}{\epsilon_0} q_i\right).$$

Thus for any closed surface, we can write

$$\boxed{\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}},}$$

where Q_{enc} is the total charge enclosed within the surface.

3. Gauss's Law in differential form :

By applying divergence theorem we can have,

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau.$$

Also Q_{enc} can be expressed in terms of the charge density ρ, and we have write

$$Q_{\text{enc}} = \int_{\mathcal{V}} \rho\, d\tau.$$

So Gauss's law becomes

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau = \int_{\mathcal{V}} \left(\frac{\rho}{\epsilon_0}\right) d\tau.$$

As dτ is arbitrary, i.e., this holds for any volume, the integrands must be equal and can be written as

$$\boxed{\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho.}$$

It is the Gauss's law in differential form.

4. Divergence of the electrostatic field :

We already know that

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}\, d\tau'.}$$

Note that the integration was over the volume occupied by the charge originally, but since ρ = 0 in the exterior region anyway. Therefore, we may write

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \rho(\mathbf{r}')\, d\tau'.$$

Now

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \rho(\mathbf{r}')\, d\tau'.$$

We know that,

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = 4\pi\, \delta^3(\boldsymbol{\mathscr{r}}).$$

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int 4\pi\, \delta^3(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}')\, d\tau' = \frac{1}{\epsilon_0} \rho(\mathbf{r}),$$

This is Gauss's law in differential form as derived above.

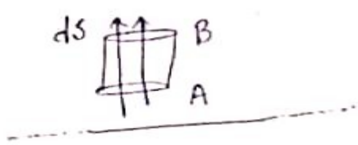
In a reverse, we can reach the original form of Gauss's law as given below :

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E}\, d\tau = \oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho\, d\tau = \frac{1}{\epsilon_0} Q_{\text{enc}}.$$

5. Some applications of Gauss theorem and Coulomn's theorem :

(A) Intensity at a point near an infinite sheet of charge/charged infinite plane:

We consider an infinite sheet of Charge having an uniform surface density



σ (considering both surfaces). The electric field intensity at any point near the surface is required.

Let us consider two points A & B near the infinite sheet. Let E and E' be the electric field intensities at A & B respectively. We describe an elementary pill-box such that the points A and B lie on the flat surfaces of the pill box.

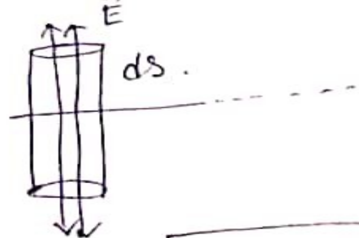
Since, the sheet is of infinite dimension, the lines of force will be perpendicular to the flat surfaces and parallel to the curved surface of the pill-box. If "ds" be the area of each flat surface, the total normal outward flux from the pill-box will be,

$$\phi = -E\,ds + E'ds$$

Since, there is no charge inside the pill box $\Rightarrow \phi = 0$

$\Rightarrow E' = E$

Now assume another pill-box with flat surfaces on either side of the sheet. Here



$$\phi = E\,ds + E\,ds = 2E\,ds$$

$\Rightarrow$ From Gauss theorem, $2E\,ds = \dfrac{\sigma\,ds}{\epsilon_0} \Rightarrow \boxed{E = \dfrac{\sigma}{2\epsilon_0}}$

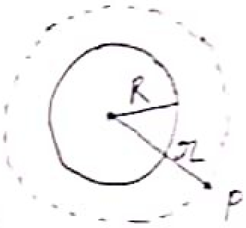
(B) Uniformly charged sphere:

Consider a sphere of radius R having uniform volume density of charge ρ.

$\therefore$ Total charge $Q = \frac{4}{3}\pi R^3 \rho$

(i) <u>Field outside the sphere</u> $\Rightarrow$

Let us draw a concentric sphere with radius "r" such that the point "P", where the field to be measured, lies on the surface.



Thus the total outward normal flux from the sphere of radius "r"

$$\phi = E\,(4\pi r^2)$$

From Gauss's theorem $\Rightarrow$ $E.\,4\pi r^2 = \dfrac{\sigma\, 4\pi R^2}{\epsilon_0}$

or, $\quad E.\,4\pi r^2 = \dfrac{\frac{4}{3}\pi R^3 \rho}{\epsilon_0}$ (in terms of ρ)

$\therefore$ In terms of Q_{enc} $\Rightarrow$ we will obtain

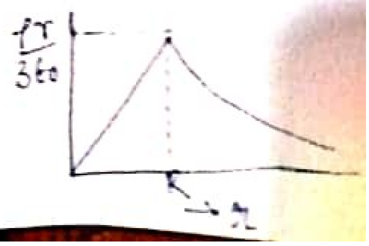
$$\boxed{E = \frac{Q}{4\pi\epsilon_0 r^2}} \quad \text{or} \quad \boxed{E = \frac{R^3 \rho}{3\epsilon_0 r^2}}$$

(ii) <u>Field inside the sphere</u> $\Rightarrow$

Now P is within the sphere & $r < R$.

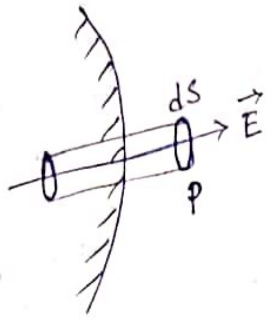
Here $\quad (4\pi r^2)\,E = \left(\frac{4}{3}\pi r^3 . \rho\right) / \epsilon_0$

$$\Rightarrow \quad \boxed{E = \frac{\rho r}{3\epsilon_0}}$$



(C) Charged conductor:

We consider a charged conductor having an uniform surface charge density σ.



Again, we describe an elementary pill-box such that 'P' lies on the flat surface as depicted.

$\therefore \quad \phi = E\,ds$.

Again from the Gauss's theorem $\Rightarrow$

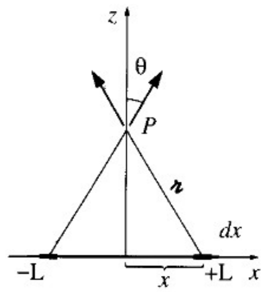
$$\phi = E\,ds = \frac{\sigma\,ds}{\epsilon_0}$$

$$\Rightarrow \boxed{E = \frac{\sigma}{\epsilon_0}}$$

(D) Infinite line of charge :

For simplicity, tet us chop the line symmetrically in two halves (total length 2L) , so that the horizontal components of the two fields cancel, and the net field of the pair can be written as,

$$d\mathbf{E} = 2\frac{1}{4\pi\epsilon_0}\left(\frac{\lambda\,dx}{r^2}\right)\cos\theta\,\hat{\mathbf{z}}.$$



Here $\cos\theta = z/r$, $r = \sqrt{z^2 + x^2}$, and x runs from 0 to L:

$$\begin{aligned} E &= \frac{1}{4\pi\epsilon_0}\int_0^L \frac{2\lambda z}{(z^2+x^2)^{3/2}}dx \\ &= \frac{2\lambda z}{4\pi\epsilon_0}\left[\frac{x}{z^2\sqrt{z^2+x^2}}\right]\Bigg|_0^L \\ &= \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}, \end{aligned}$$

and it aims in the z-direction.

For points far from the line ($z \gg L$), this result simplifies:

$$E \cong \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z^2},$$

For infinitely long wire ($L \rightarrow \infty$)

$$E = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{z};$$

or, more generally,

$$E = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{s},$$

where s is the distance from the wire.

Home Work : Try Electric field due to point charge and Hollow Sphere .

Charge density on the surface of a conductor:

Properties of a conductor $\Rightarrow$

1. In a charged conductor, under static condition, the potential must be the same at all points $\Rightarrow$ Electric field in a conductor is zero.

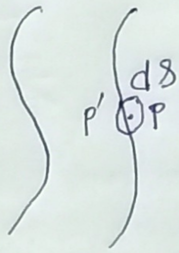
2. The net charge of a charged conductor resides on the outer surface. Since $\vec{E} = 0$ & $\nabla \cdot \vec{E} = \rho/\epsilon_0$
$\Rightarrow$ charge density inside the conductor $\rho = 0$

3. Under static condition, the electric field just outside the charged conductor is perpendicular to the surface of the conductor, creating a non-static condition.

4. For such case, we already have proved $\boxed{E = \frac{\sigma}{\epsilon_0}}$

Force per unit area on the surface :

- The equation $E = \frac{\sigma}{\epsilon_0}$ gives the electric field at any point P, but outside the surface of the charged conductor.
- We may write $E = E_1 + E_2$,
  $E_1 \rightarrow$ field due to the charge on a tiny area ds around P,
  $E_2 \rightarrow$ due to the charge on rest of the surface.



- Consider a point P' within the conductor, very near to "ds", but on the opposite side of P. The intensity at P' due to charge on ds will be equal to E_1 but opposite in direction, whereas E_2 will remain the same.
- But inside the conductor $E = 0 \Rightarrow -E_1 + E_2 = 0$
  or, $E_2 = E_1 = \frac{E}{2} = \frac{\sigma}{2\epsilon_0}$
- The force on a charge is given by the product of the charge and the intensity due to other charges at the ~~position~~ position of the said charge. Thus the force on σds is

$$F = (\sigma ds) . E_2 = \sigma ds . \frac{\sigma}{2\epsilon_0} = \frac{\sigma^2 ds}{2\epsilon_0}$$

$\Rightarrow$ Pressure or force/unit area $\Rightarrow$

$$\boxed{\frac{F}{ds} = \frac{\sigma^2}{2\epsilon_0}}$$ in $N m^{-2}$ unit

CURL OF ELECTROSTATIC FIELD & ITS NATURE

Let us start from the easiest case ⇒

For a point charge ⇒ $\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$

- Line integral of $\vec{E}$ from 'a' to 'b' in a trajectory can be written as ⇒

$$\int_a^b \vec{E}\cdot\vec{dl}$$, $\vec{dl}$ is the line element along the path.

- In spherical co-ordinate ⇒

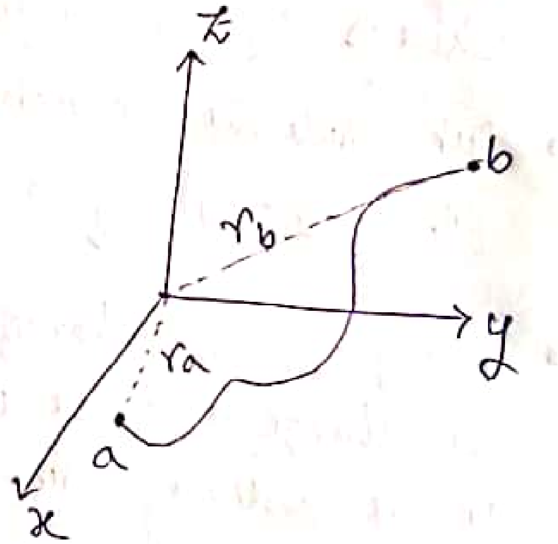
$$\vec{dl} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$\Rightarrow \int_a^b \vec{E}\cdot\vec{dl} = \frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2}\,dr = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_a} - \frac{1}{r_b}\right]$$

- For a closed contour ⇒

$\boxed{a=b} \Rightarrow \boxed{r_a = r_b}$

$$\Rightarrow \boxed{\oint \vec{E}\cdot\vec{dl} = 0}$$



- Now applying Stokes' theorem ⇒

$\boxed{\vec{\nabla}\times\vec{E} = 0}$ ⇒ This equation holds for any static distribution whatever.

⇒ It confirms the conservative nature of electrostatic field.

INTRODUCTION TO ELECTRIC POTENTIAL

As $\vec{\nabla} \times \vec{E} = 0$, $\Rightarrow$ ($\vec{\nabla} \times \vec{E}$ - is the curl)

$\therefore \oint \vec{E} \cdot d\vec{l} = 0$

Let assume one goes out along the path (i) and returns along path (ii), still over the closed contour we can have $\oint \vec{E} \cdot d\vec{l} = 0 \Rightarrow$ This is a path independant line integral.

Let us define a function $V(r) = -\int_{O}^{r} \vec{E} \cdot d\vec{l}$

$\Rightarrow$ It is called <u>electric potential</u>.

Here $O \rightarrow$ assumed as a standard reference point.

$\therefore$ Potential difference between point A & B

$$V(B) - V(A) = -\int_{O}^{B} \vec{E} \cdot d\vec{l} + \int_{O}^{A} \vec{E} \cdot d\vec{l} = -\int_{A}^{B} \vec{E} \cdot d\vec{l}$$

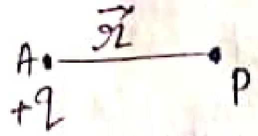
Again, from fundamental theorem of gradients, we can have $\Rightarrow$ $V(B) - V(A) = \int_{A}^{B} (\vec{\nabla} V) \cdot d\vec{l}$

Equating these two equations (as dl is arbitrary)

$$\boxed{\vec{E} = -\vec{\nabla} V}$$

Thus, by knowing the potential V, one can calculate $\vec{E}$ using the above relation.

Let us now set the reference point O to ∞ (infinity). Then the electric potential at P due to a point charge "+q" at point A can be defined as work done in bringing an unit positive charge from infinite to the said point.



$$\Rightarrow \quad V(r) = -\int_{\infty}^{r} \vec{E}\cdot\vec{dl} = \frac{1}{4\pi\epsilon_0}\int_{r}^{\infty} \frac{q}{r'^2}\, dr'$$

$$\Rightarrow \quad \boxed{V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}}$$

∴ If there exist a collection of discrete charges, using superposition theory ⇒

$$\boxed{V(r) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}}$$

For continuous distribution ⇒

$$\boxed{V(r) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{r}\, dq}$$

∴ In terms of λ, σ, and ρ ⇒

$$V(r) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')}{r}\, d\tau' \quad \text{(for volume charge distribution)}$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(r')}{r}\, dl' \quad \text{(for line charge distribution)}$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(r')}{r}\, da' \quad \text{(for surface charge distribution)}$$

POTENTIAL OF UNIFORMLY CHARGED SPHERICAL SHELL

For surface charge distribution $\sigma \Rightarrow$

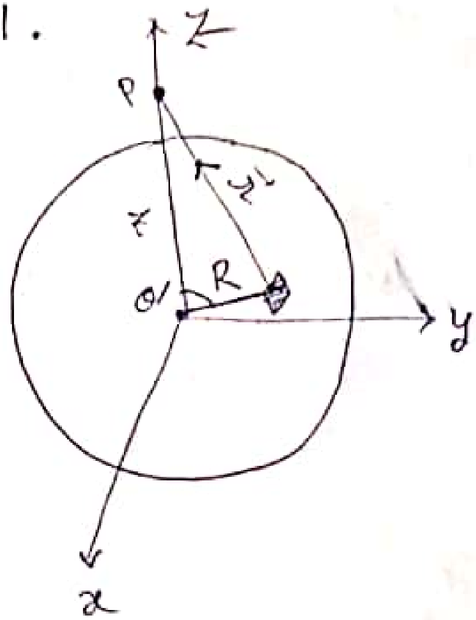
$$V(r) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{\mathscr{r}}\, da'$$, da' is an infinitesimally small surface element

If $R \to$ radius of the shell.

From the diagram $\Rightarrow$

$$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta'$$

& $$da' = R^2 \sin\theta' d\theta' d\phi'$$



$\Rightarrow V(r)$

$$= \frac{\sigma}{4\pi\epsilon_0}\int \frac{R^2 \sin\theta' d\theta' d\phi'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}$$

$$= \frac{2\pi R\sigma}{z}\left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right]$$ (by solving $\theta \to 0$ to π)

If P is outside $\Rightarrow z > R$

$\therefore \sqrt{(R-z)^2} = (z-R)$

$$\boxed{\Rightarrow V(z) = \frac{R^2\sigma}{\epsilon_0 z}}$$

At this stage we must be careful to take +ve root

& P is inside $\Rightarrow z < R$

$\therefore \sqrt{(R-z)^2} = (R-z)$

$$\Rightarrow \boxed{V(z) = \frac{R\sigma}{\epsilon_0}}$$

[Home work: Calculate the potential for solid sphere]