Surendranath Evening College

Department of Physics

WAVES

Topics :: 1. Oscillations and 2. Superposition of Harmonic Oscillations

PHSA-CC-2-4-Th

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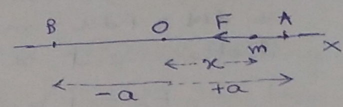
Oscillations

SIMPLE HARMONIC MOTION

Simple harmonic motion (S.H.M.) is a special type of periodic motion in which the restoring force acting on the particle is proportional to the displacement from the mean position and is always directed to the mean position.

A system executing S.H.M. is called harmonic oscillator.



Let us consider a particle of mass m, executing s.h.m. about the equilibrium position O with maximum displacement a on either side. At any instant t, if the displacement of the particle is x, then acceleration will be d^2x/dt^2.
Now according to the definition of S.H.M.

Restoring force (F) $\propto$ − displacement (x)

or, $m\frac{d^2x}{dt^2} \propto -x$ or, $m\frac{d^2x}{dt^2} = -kx$

or, $m\frac{d^2x}{dt^2} + \omega^2 x = 0$ .... ①

k is force constant. negative sign shows that restoring forces is always opposite to x. let $\omega^2 = \frac{k}{m}$.

It is the differential equation of SHM.
Multiplying equation ① by $2\frac{dx}{dt}$ and integrating we obtain

$\left(\frac{dx}{dt}\right)^2 + \omega^2 x^2 = c$ ..... ②

c = constant of integration.

We know velocity (dx/dt) of particle is zero at maximum value of displacement i.e.
at $x = a$, $dx/dt = 0$. Putting these values in eqn. (2) we get $c = \omega^2 a^2$ and then substituting the value of c in eqn. (2) we get

$(dx/dt)^2 + \omega^2 x^2 = \omega^2 a^2$ or, $(dx/dt)^2 = \omega^2(a^2 - x^2)$

or, $dx/dt = \pm\,\omega\sqrt{a^2 - x^2}$ .... (3)

Taking the positive root and integrating we get

$\sin^{-1}\left(\frac{x}{a}\right) = \omega t + \phi$ or, $x = a\sin(\omega t + \phi)$ ---- (4)

Again taking negative root from eqn. (3) we get after integration

$\cos^{-1}\left(\frac{x}{a}\right) = \omega t + \phi$ or, $x = a\cos(\omega t + \phi)$ ------ ⑤

Both equations ⑤ and ⑥ are general solutions of eqn. ② where a = amplitude and ϕ is initial phase. Both are valid solutions. Some quantities related to ~~eqn.~~ solution ④ of SHM is given below:—

ⓐ Displacement $x = a\sin(\omega t + \phi)$

ⓑ velocity $v = dx/dt = \omega a\cos(\omega t + \phi) = \omega a\sqrt{a^2 - x^2}$ .... (5A)

ⓒ Acceleration $(f) = -\omega^2 x$

ENERGY OF A PARTICLE EXECUTING SHM

A particle in SHM has two types of energy:-
(a) Potential energy due to displacement from mean position
(b) Kinetic energy due to its velocity.

Work done for small displacement dx against force is $dW = F dx = m\omega^2 x\, dx$. Then total work done for displacement x is potential energy (E_P).

$$\therefore E_P = \int_0^x m\omega^2 x\, dx = \frac{1}{2} m\omega^2 x^2 \quad \ldots\ldots (6)$$

Velocity of the particle for displacement x is $v = \omega\sqrt{a^2 - x^2}$

$$\therefore \text{Kinetic energy } (E_K) = \frac{1}{2} m v^2 = \frac{1}{2} m\omega^2 (a^2 - x^2) \quad \ldots (7)$$

$$\therefore \text{Total mechanical energy} (E) = E_K + E_P = \frac{1}{2} m\omega^2 a^2 \quad \ldots (8)$$

So total mechanical energy of the particle in SHM is constant and does not depend on displacement.

Time average of Kinetic and Potential Energies

The time average of a quantity $Q(t)$ over a time interval T is given by

$$Q_{av} = \frac{1}{T}\int_0^T Q(t)\, dt$$

Hence the average K.E. for a period T is given by

$$(K.E.)_{av} = \frac{1}{T}\int_0^T E_K\, dt = \frac{1}{T}\int_0^T \frac{1}{2} m v^2\, dt = \frac{1}{T}\int_0^T \frac{1}{2} m\omega^2 a^2 \cos^2(\omega t + \phi)\, dt$$

$$= \frac{m a^2\omega^2}{2T}\int_0^{T = \frac{2\pi}{\omega}} \frac{1}{2}[1 + \cos(2\omega t + 2\phi)]\, dt = \frac{m a^2 \omega^2}{4(2\pi/\omega)} \cdot \frac{2\pi}{\omega} = \frac{m a^2\omega^2}{4}$$

Similarly, $(P.E.)_{av} = \frac{1}{T}\int_0^T (E_P)\, dt = \frac{1}{T}\int_0^T \frac{1}{2} m\omega^2 \sin^2(\omega t + \phi)\, dt$

$$= \frac{1}{4} m a^2 \omega^2$$

So we see that $(K.E.)_{av} = (P.E.)_{av}$

Also, $(K.E.)_{av} + (P.E.)_{av} = \frac{1}{4} m a^2\omega^2 + \frac{1}{4} m a^2 \omega^2 = \frac{1}{2} m a^2 \omega^2$

$= E$ (total mechanical energy)

DAMPED OSCILLATION

When there is no dissipative forces, a body continues to oscillate for indefinite time with constant amplitude. Such an ideal vibration is called free vibration. The frequency of such free vibration is characteristics of the system and is called the natural frequency of the system.

However, ~~there a~~ dissipative forces are always acting on real physical system which causes the system to lose energy with time. So we may conclude that there is a damping force on a vibrating body due to friction, viscosity of the medium ~~et~~ and other effects. For example, the amplitude of oscillation of a simple pendulum in air decreases with time and ultimately comes to rest. This damping force always acts in opposition to the motion.

Mathematical Analysis of Damped Vibration

Suppose a particle of mass m is oscillating along x axis. In simplest case, the damping forces may be taken to be proportional to the instantaneous velocity of the moving body. Also, there is a restoring force proportional to displacement.

So the equation of motion may be written as

$$m\frac{d^2x}{dt^2} = -R_m\frac{dx}{dt} - sx \quad \text{......} \quad (1)$$

where, R_m = damping force per unit velocity

s = restoring force per unit displacement.

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0 \quad — (2) \quad \text{where, } 2b = \frac{R_m}{m} \ \& \ \omega_0 = \sqrt{\frac{s}{m}}$$

ω_0 may be thought ~~that~~ as angular frequency in absence of damping.

To solve eqn. (2) we take $x = ce^{\alpha t}$ be a trial soln.

where c and α are arbitrary constants. Differentiating we get, $\frac{dx}{dt} = \alpha c e^{\alpha t}$ and $\frac{d^2x}{dt^2} = \alpha^2 c e^{\alpha t}$ ~~and hence~~

So, from eqn. (2) we get $(\alpha^2 + 2b\alpha + \omega_0^2)ce^{\alpha t} = 0$ — (3)

If eqn. (3) to be valid for all t we can write

$$\alpha^2 + 2b\alpha + \omega_0^2 = 0$$

It has two roots, $\alpha_1 = -b + \sqrt{b^2 - \omega_0^2}$; $\alpha_2 = -b - \sqrt{b^2 - \omega_0^2}$

So the general solution would be $x = c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t}$

$$x = e^{-bt}\left[c_1 e^{\sqrt{b^2-\omega_0^2}\,t} + c_2 e^{-\sqrt{b^2-\omega_0^2}\,t}\right] \quad — (4)$$

where c_1 and c_2 are arbitrary constants. Depending on damping factor b, the solution takes three distinct forms :-

Case 1: Large damping i.e. $b > \omega_0$ (Overdamped case)

If the damping is very large i.e. $b > \omega_0$ then $\sqrt{b^2-\omega_0^2}$ is a positive quantity, so the displacement x quickly falls to zero and no vibration is executed. The displacement reaches a maximum value and diminishes exponentially. Here the motion is non-oscillatory. This is overdamped or dead-beat motion as found in dead-beat moving coil galvanometer.

Case II: Critical damping i.e. $b = \omega_0$

In this case displacement equation takes the form $x = (c_1 + c_2 t) e^{-\omega_0 t}$ --- (5)

This motion is also non-oscillatory but the rate of decay is much faster than the overdamped case.

The motion is now said to be critically damped.

Case III: Small damping $b < \omega_0$

The most important type of behaviour is obtained when $b < \omega_0$ i.e. ~~small~~ damping is low. In this case $b^2 - \omega_0^2$ is negative and we can write

$\sqrt{b^2-\omega_0^2} = \sqrt{-(\omega_0^2 - b^2)} = j\omega$ where $\omega = \sqrt{\omega_0^2 - b^2}$

The solution (4) takes the form

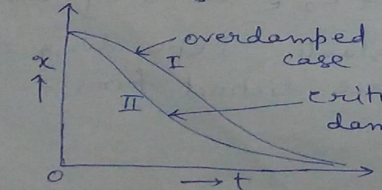
$$x = e^{-bt}[c_1 e^{j\omega t} + c_2 e^{-j\omega t}]$$
$$= e^{-bt}[c_1(\cos\omega t + j\sin\omega t) + c_2(\cos\omega t - j\sin\omega t)]$$
$$= e^{-bt}[(c_1+c_2)\cos\omega t + j(c_1-c_2)\sin\omega t]$$
$$= e^{-bt}[B_1\cos\omega t + B_2 \sin\omega t]$$
$$= A e^{-bt}\cos(\omega t - \phi)$$

Putting $B_1 = A\cos\phi = c_1 + c_2$, $B_2 = j(c_1 - c_2) = A\sin\phi$

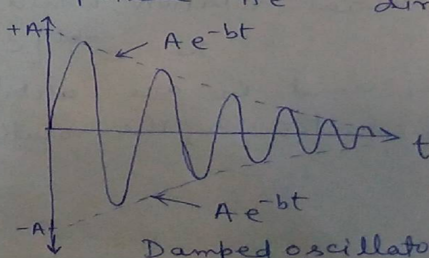
where A and ϕ are constants which can be evaluated from initial conditions.

The equation $x = A e^{-bt}\cos(\omega t - \phi)$ --- (6)

represents a damped oscillatory motion with an angular frequency $\omega = \sqrt{\omega_0^2 - b^2}$. The amplitude Ae^{-bt} diminishes exponentially with time.



Non-oscillatory behaviour



Damped oscillatory motion. (Case III)

CHARACTERIZING WEAK DAMPING

The following three parameters characterize the weak damping:

a) Relaxation time (τ) or decay constant
b) Logarithmic decrement (λ)
c) Quality factor

a) Relaxation time (τ) or decay constant

It refers to the time in which the amplitude of a weakly damped system reduces to $1/e$ times of the original value. In other words, it is the time in which the mechanical energy of an oscillator decays to $1/e$ times its initial value.

At $t = \tau$, $A = \frac{A_0}{e}$ or $E = \frac{E_0}{e}$. Here τ comes as $\frac{1}{b}$

b) Logarithmic Decrement

Let at time $t = t_1$ amplitude $A_1 = A e^{-bt_1}$. Again let after $T/2$ (i.e. half time period later) the amplitude $A_2 = A e^{-b(t_1 + \frac{T}{2})}$
$= A_1 e^{-\frac{bT}{2}}$

Thus, $\frac{A_1}{A_2} = e^{bT/2} = d$ (say any constant)

or, $\log_e d = \frac{bT}{2} = \lambda$ (say)

The quantity d is the ratio of two successive amplitudes on opposite side of mean position. d refers to the decrease in successive amplitude and is called the decrement. ~~$\lambda = d$~~ $\lambda = \log_e d$ is called the logarithmic decrement of the damped oscillation.

c) Quality Factor

Quality factor measures the quality of harmonic oscillator as far as damping is concerned. "Lesser the damping, better will be the quality of harmonic oscillator as an oscillator" The effect of damping on the motion of the oscillator can be expressed in terms of quality factor. It is defined as

$$Q = 2\pi \times \frac{\text{Average energy stored per cycle}}{\text{Average energy lost per cycle}}$$

In this case Q comes out as $\frac{\omega}{2b}$.

Forced Vibration

The natural vibration of a system dies out with time due to damping forces. But the vibration can be maintained by applying external periodic forces. In presence of external periodic force, initially the system vibrates with both frequencies. Finally in steady state, the ~~natur~~ natural vibration dies out due to damping and the system vibrates with the frequency of driving force. Such vibrations are called forced vibration.

Let us ~~suppose~~ consider a system of mass m where restoring force and damping forces are present. An external periodic force $F = F_0 \sin \omega t$ is applied on the system. The equation of motion can be written as

$$m\frac{d^2x}{dt^2} + R_m\frac{dx}{dt} + Sx = F_0 \sin \omega t$$

or, $$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = f_0 \sin \omega t \quad \text{------ (1)}$$

where, R_m = damping constant, S = restoring force

$2b = \frac{R_m}{m}$; $\omega_0^2 = \frac{S}{m}$; $f_0 = \frac{F_0}{m}$; ω_0 = natural frequency, ω = frequency of driving force

Let $x = A\sin(\omega t - \theta)$ is the steady state solution. ---(2)

$\therefore \frac{dx}{dt} = \omega A\cos(\omega t-\theta)$ and $\frac{d^2x}{dt^2} = -\omega^2 A\sin(\omega t-\theta)$

Substituting these values in eqn. (1) we get

$$-\omega^2 A\sin(\omega t-\theta) + 2b\omega A\cos(\omega t-\theta) + \omega_0^2 A\sin(\omega t-\theta) = f_0 \sin(\omega t-\theta+\theta)$$

or, $$A(\omega_0^2-\omega^2)\sin(\omega t-\theta) + 2b\omega A\cos(\omega t-\theta)$$
$$= f_0\cos\theta \sin(\omega t-\theta) + f_0 \sin\theta\cos(\omega t-\theta) \quad \text{---- (3)}$$

So we can write comparing both sides of eqn. (3)

$$2b\omega A = f_0 \sin\theta \quad \text{and} \quad (\omega_0^2-\omega^2)A = f_0\cos\theta$$

Squaring and adding these two expressions

$$4b^2\omega^2A^2 + (\omega_0^2-\omega^2)^2A^2 = f_0^2(\sin^2\theta + \cos^2\theta)$$

$$\therefore A^2 = \frac{f_0^2}{(\omega_0^2-\omega^2)^2 + 4b^2\omega^2} \quad \text{---- (3A)}$$

and $$\tan\theta = \frac{2b\omega}{\omega_0^2-\omega^2}$$

So the general solution becomes from equation (2)

$$x = \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2+4b^2\omega^2}}\sin\left(\omega t - \tan^{-1}\frac{2b\omega}{\omega_0^2-\omega^2}\right) \quad \text{---- (4)}$$

$$= A\sin(\omega t-\theta)$$

This equation gives the forced vibration of the system in steady state. This is a particular solution called particular integral.

Early stage :— The external force when just begins to act on the damped system sets up an impulse and the consequent natural vibration is represented by

$m\frac{d^2x}{dt^2} + R\frac{dx}{dt} + Sx = 0$ and has solution

$x = A_1 e^{-bt} \cos(\sqrt{\omega_0^2 - b^2}\, t - \delta)$ where A_1 and δ are arbitrary constants.

So the general solution is represented by

$$x = A_1 e^{-bt} \cos(\sqrt{\omega_0^2 - b^2}\, t - \delta) + A \sin(\omega t - \theta) \quad \text{— (5)}$$

It is evident that due to damping (e^{-bt}) the first term of eqn. (5) becomes negligible very soon and free oscillation die out. This first term of equation (5) gives the natural vibration of the particle at begining. Second term represents forced vibration due to external driving force which maintains the vibration. At initial stage both terms are present. First term is called "transient term". Finally only the steady state term stays.

Amplitude Resonance

Resonance is a special case of forced vibration. If the frequency of the driving force agrees with the natural frequency of the system then the system oscillates with maximum amplitude or velocity. This is called resonance where driving force transfers maximum power or energy to the system.

From steady state solution (4), it is seen that amplitude A of forced vibration depends on frequency ω of the driving force. Amplitude becomes maximum for some values of ω and it is called amplitude resonance between driver and driven system. For the amplitude A to be maximum we must have

$\frac{dA}{d\omega} = 0$ and $\frac{d^2A}{d\omega^2} > 0$ [from expression(3A)]

From eqn. (3A) we can write

$$\frac{d}{d\omega}\{(\omega_0^2 - \omega^2)^2 + 4b^2\omega\} = 0$$

or, $2(\omega_0^2-\omega^2)(-2\omega) + 8b^2\omega = 0$

or, $\omega = \sqrt{\omega_0^2 - 2b^2} = \omega_r$ (say) ——— (6)

* We can also show that $\frac{d^2A}{d\omega^2} > 0$ at $\omega = \omega_r$.

So ω_r is the frequency at which amplitude resonance occurs. Maximum amplitude at resonance is obtained by putting $\omega = \sqrt{\omega_0^2 - 2b^2}$ in equation (3A)

So,

$$A_{max} = \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2 + 4b^2\omega^2}} = \frac{f_0}{\sqrt{(2b^2)^2 + 4b^2\omega^2}}$$

or,

$$A_{max} = \frac{f_0}{\sqrt{4b^4 + 4b^2\omega^2}} = \frac{f_0}{\sqrt{4b^2(b^2+\omega^2)}} = \frac{f_0}{2b\sqrt{b^2+\omega_0^2}}$$

$$= \frac{f_0}{2b\sqrt{\omega_0^2 - b^2}} \quad \cdots (6A)$$

[As from eqn. (6) $\omega^2 + b^2 = \omega_0^2 - b^2$]

When there is no damping ($b = 0$) & $\omega = \omega_0$ from expression (6) amplitude becomes maximum. Such a case is never realised in practice. However if b is small compared to ω_0, then $\omega \approx \omega_0$ (from expression (6)) and we get maximum amplitude $A_{max} = \frac{f_0}{2b\omega_0}$ [using expression (6A)]

So amplitude is practically greatest at $\omega = \omega_0$ and

$$A_{max} = \frac{f_0}{2b\omega}$$

2.5 Superposition of two SHMs acting at right angles to each other :

(a) Oscillations having same frequencies :

Let two simple harmonic motions acting at right angles be represented by the equations

$$x = a\cos(\omega t + \delta_1) \qquad \text{...(2.5-1)}$$

and

$$y = b\cos(\omega t + \delta_2) \qquad \text{...(2.5-2)}$$

The resulting motion can be obtained by eliminating t from the Eqs. (2.5-1) and (2.5-2). From Eq. (2.5-2) we can write

$$\frac{y}{b} = \cos(\omega t + \delta_1 + \delta_2 - \delta_1) = \cos(\omega t + \delta_1 + \delta)$$

where $\delta = \delta_2 - \delta_1$ is the phase difference between the two SHMs.

Therefore, $\frac{y}{b} = \cos(\omega t + \delta_1).\cos\delta - \sin(\omega t + \delta_1).\sin\delta$

Using Eq. (2.5-1) we can write

$$\frac{y}{b} = \frac{x}{a}\cos\delta - \sqrt{1 - x^2/a^2}.\sin\delta$$

or,

$$\left(\frac{x}{a}\cos\delta - \frac{y}{b}\right)^2 = \left(1 - \frac{x^2}{a^2}\right)\sin^2\delta$$

or,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\delta = \sin^2\delta \qquad \text{...(2.5-3)}$$

It represents the general equation of an ellipse bounded within a rectangle of sides $2a$ and $2b$. Thus the resultant motion is in general elliptic. Let us consider a few special cases :

(i) If $\delta = \delta_2 - \delta_1 = 0$ *i.e.*, the two SHMs are in phase, then the Eq. (2.5-3) reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0$$

or,

$$\left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0$$

It represents a pair of coincident straight lines $y = \frac{b}{a}x$, passing where $A^2 = \left(\sum_i a_i \sin\delta_i\right)^2 + \left(\sum_i a_i \cos\delta_i\right)^2$ and $\tan\phi = \dfrac{\sum_i a_i \sin\delta_i}{\sum_i a_i \cos\delta_i}$

Vector method :

The rotating vector representation of SHM provides a simple method of obtaining the resultant of SHMs of same frequency. The SHM of Eq. (2.4-1) can be represented by a rotating vector $\overrightarrow{OB_1}$ of constant length a_1 rotating anti-clockwise with a constant angular velocity ω and making an angle $\omega t + \delta_1$ with the x–axis at any time t. The projection ON_1 of this vector on x–axis gives the displacement x_1 at any time t. Similarly the rotating vector $\overrightarrow{OB_2}$ in Fig. 2.4-1 will represent the SHM of Eq. (2.4-2). Now the resultant motion will be given by the vector sum of $\overrightarrow{OB_1}$ and $\overrightarrow{OB_2}$.

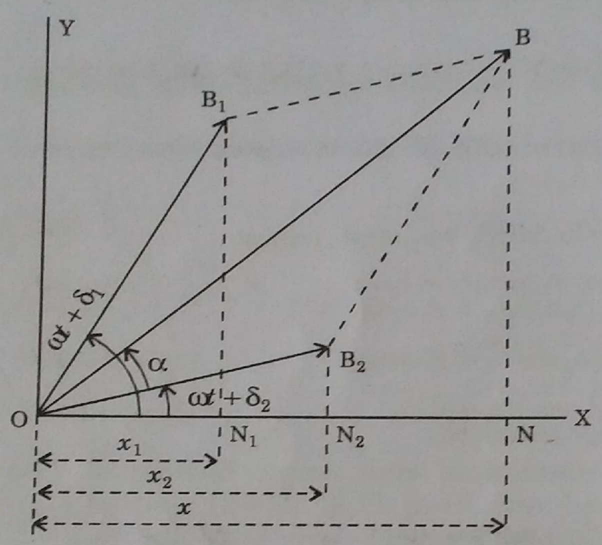


Fig. 2.4-1

By parallelogram law of vector addition the magnitude A of the resultant $\overrightarrow{OB}$ is given by

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\delta_1 - \delta_2)$$

which is the same as obtained earlier.

If the resultant $\overrightarrow{OB}$ makes an angle $\omega t + \delta$ with x-axis, then from Fig. 2.4-1,

$$\delta = \delta_2 + \alpha$$

$\therefore$
$$\tan\delta = \frac{\tan\delta_2 + \tan\alpha}{1 - \tan\delta_2 \tan\alpha}$$

Now,
$$\tan\alpha = \frac{a_1 \sin(\delta_1 - \delta_2)}{a_2 + a_1\cos(\delta_1 - \delta_2)}$$

Substituting $\tan\alpha$ in the above equation and simplifying we get

$$\tan\delta = \frac{a_1 \sin\delta_1 + a_2 \sin\delta_2}{a_1 \cos\delta_1 + a_2 \cos\delta_2}$$

which is the same as obtained earlier.

The projection of $\overrightarrow{OB}$ on x-axis is

$$x = A\cos(\omega t + \delta)$$

which represents a SHM.

(b) Two SHMs of slightly different frequencies acting along the same direction : Beats :

Let us consider two SHMs having slightly different angular frequencies ω and $\omega + \Delta\omega$ where $\Delta\omega \ll \omega$.

$$x_1 = a_1 \cos(\omega t + \delta_1) \qquad \text{...(2.4-8)}$$

$$x_2 = a_2 \cos[(\omega + \Delta\omega)t + \delta_2] \qquad \text{...(2.4-9)}$$

$$= a_2 \cos(\omega t + \delta_2')$$

where $\delta_2' = \Delta\omega t + \delta_2$.

The resultant displacement is given by

$$x = x_1 + x_2$$

$$= (a_1 \cos\delta_1 + a_2 \cos\delta_2')\cos\omega t - (a_1 \sin\delta_1 + a_2 \sin\delta_2')\sin\omega t$$

Putting $\qquad a_1 \cos\delta_1 + a_2 \cos\delta_2' = A\cos\phi \qquad$ ...(2.4-10)

and $\qquad a_1 \sin\delta_1 + a_2 \sin\delta_2' = A\sin\phi \qquad$ ...(2.4-11)

we get, $\qquad x = A\cos(\omega t + \phi) \qquad$ ...(2.4-12)

where $\qquad A^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos(\delta_1 - \delta_2')$

$$= a_1^2 + a_2^2 + 2a_1 a_2 \cos(\Delta\omega t + \delta_2 - \delta_1) \qquad \text{...(2.4-13)}$$

and $\qquad \tan\phi = \dfrac{a_1 \sin\delta_1 + a_2 \sin\delta_2'}{a_1 \cos\delta_1 + a_2 \cos\delta_2'} \qquad$ ...(2.4-14)

The resultant motion described Eq. (2.4-12) is not simple harmonic because here both the amplitude A and the phase constant ϕ vary with time. As the time increases the amplitude A attains maximum value $a_1 + a_2$ when $\cos(\Delta\omega t + \delta_2 - \delta_1) = 1$ or, $\Delta\omega t + \delta_2 - \delta_1 = 0, 2\pi, 4\pi, \ldots$ etc.

The amplitude attains minimum value $a_1 \sim a_2$ when $\cos(\Delta\omega t + \delta_2 - \delta_1) = -1$ or, $\Delta\omega t + \delta_2 - \delta_1 = \pi, 3\pi, 5\pi, \ldots$ etc.

Hence as the time goes on the amplitude of the resultant motion passes alternatively through maximum and minimum. The time interval between two successive maxima or minima of amplitude is obviously $2\pi/\Delta\omega$. Thus the resultant amplitude varies periodically with a frequency $\Delta\omega/2\pi = \Delta\upsilon$ which is equal to the difference in frequencies of the component vibrations.

Fig. 2.4-2 displays graphically the result of superposition of two SHMs of slightly different frequencies. As an example, we show in Fig 2.4-2 superposition of two SHMs having frequencies 4 Hz and 5 Hz. They are further assumed to have same amplitudes and same initial phases.

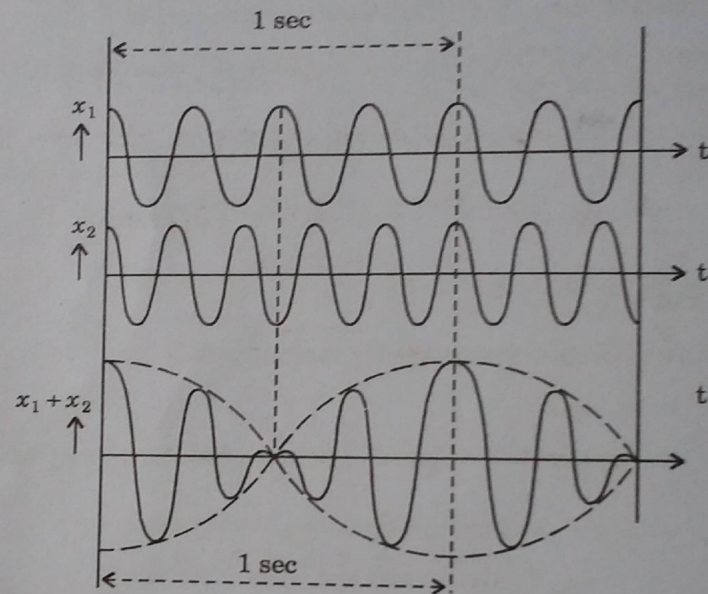


Fig. 2.4-2 : Superposition of SHMs with slightly different frequencies

It is clear from Fig. 2.4-2 that the time interval between two successive maxima or minima in the resultant pattern in this case is 1 sec. The resultant amplitude thus varies with a frequency of 1 Hz which is equal to the difference of frequencies (5 Hz - 4 Hz) of the component vibrations.

Beats :

In case of sound waves the superposition of two waves having slightly lifferent frequencies causes the intensity of the resulting sound to ncrease and decrease periodically with time. This phenomenon is known s *beats*. The number of beats per second equals the difference of requencies of the component waves.

2.5 Superposition of two SHMs acting at right angles to each other :

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and

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The resulting motion can be obtained by eliminating t from the Eqs. (2.5-1) and (2.5-2). From Eq. (2.5-2) we can write

$$\frac{y}{b} = \cos(\omega t + \delta_1 + \delta_2 - \delta_1) = \cos(\omega t + \delta_1 + \delta)$$

where $\delta = \delta_2 - \delta_1$ is the phase difference between the two SHMs.

Therefore, $\frac{y}{b} = \cos(\omega t + \delta_1).\cos\delta - \sin(\omega t + \delta_1).\sin\delta$

Using Eq. (2.5-1) we can write

$$\frac{y}{b} = \frac{x}{a}\cos\delta - \sqrt{1 - x^2/a^2}.\sin\delta$$

or,

$$\left(\frac{x}{a}\cos\delta - \frac{y}{b}\right)^2 = \left(1 - \frac{x^2}{a^2}\right)\sin^2\delta$$

or,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\delta = \sin^2\delta \qquad ...(2.5\text{-}3)$$

It represents the general equation of an ellipse bounded within a rectangle of sides $2a$ and $2b$. Thus the resultant motion is in general elliptic. Let us consider a few special cases :

(i) If $\delta = \delta_2 - \delta_1 = 0$ *i.e.*, the two SHMs are in phase, then the Eq. (2.5-3) reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0$$

or,

$$\left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0$$

It represents a pair of coincident straight lines $y = \frac{b}{a}x$, passing through the origin and inclined to the x-axis at an angle $\tan^{-1}(b/a)$ (Fig. 2.5-1).

(ii) When $\delta = \pi$, Eq. (2.5-3) becomes

$$\left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0$$

This also represents a pair of coincident straight lines passing through the origin inclined to the x-axis at an angle θ given by $\tan\theta = -b/a$.

(iii) When $\delta = \pi/2$, Eq. (2.5-3) reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

It represents an ellipse with semi-axes a and b along the coordinate axes. If in addition $a = b$ the ellipse degenerates into a circle,

$$x^2 + y^2 = a^2$$

The *direction of motion* in the elliptic or circular path can be ascertained from the equations defining the component motions. Let

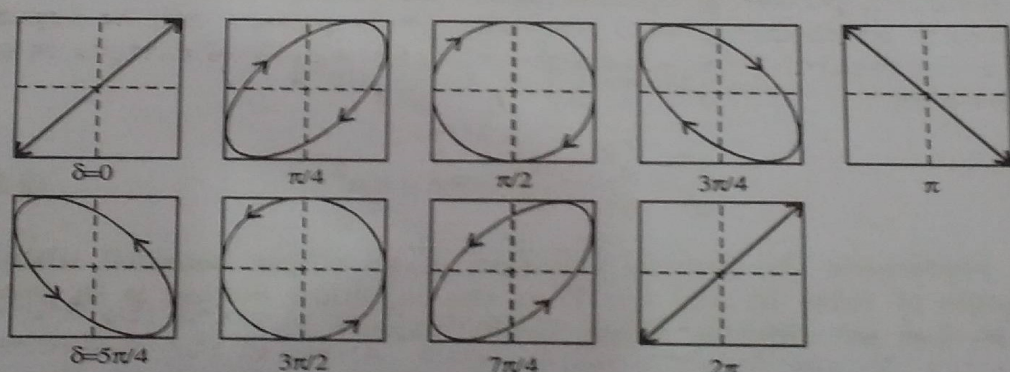

Fig. 2.5-1 : Resultant pattern due to superposition of two rectangular SHMs

us consider two rectangular SHMs of same frequency but differing in phase by $\pi/2$.

$$x = a\cos\omega t$$

$$y = b\cos(\omega t + \pi/2) = -b\sin\omega t \qquad \text{...(2.5-4)}$$

These two combine to give an elliptic motion. At time $t = 0$ the position of the point $P(x, y)$ is at $(a, 0)$. Now as t increases x decreases from its maximum value a and y begins to go negative. This indicates that the point P moves in *clockwise sense*.

Similarly we can show that the equations

$$x = a\cos\omega t$$

$$y = b\cos\left(\omega t + \tfrac{3\pi}{2}\right) = b\sin\omega t \qquad \ldots(2.5\text{-}5)$$

represent an elliptic motion in *anticlockwise sense.*

(b) Oscillations having slightly different frequencies :

Let two rectangular SHMs having slightly different frequencies be represented by

$$x = a\cos(\omega t + \delta_1) \qquad \ldots(2.5\text{-}6)$$

$$y = b\cos\left[(\omega + \Delta\omega)t + \delta_2\right] \qquad \ldots(2.5\text{-}7)$$

we can write Eq. (2.5-7) as

$$y = b\cos(\omega t + \delta_1 + \delta) \qquad \ldots(2.5\text{-}8)$$

where $\delta = \Delta\omega t + \delta_2 - \delta$ is the phase difference between the two SHMs. So proceeding as before we can show that the resultant pattern will be given by the Eq. (2.5-3). But in this case δ is not constant but varies with time. So the resultant pattern will be similar to that when the two frequencies are equal, but due to variation of phase difference δ with time the pattern changes with time. As δ changes from 0 to 2π the resultant pattern goes through all the phases shown in Fig. 2.5-1. As δ goes on changing with time the whole pattern will be repeated in a cyclic fashion. The pattern goes through a cycle of changes in time T given by

$$\delta(t+T) = \delta(t) + 2\pi$$

or,

$$\Delta\omega T = 2\pi$$

or,

$$T = \frac{2\pi}{\Delta\omega} = \frac{1}{\Delta\upsilon} \qquad \ldots(2.5\text{-}9)$$

Thus if the time T for a complete cycle is noted the frequency difference $\Delta\upsilon$ can be determined. Again, if one of the frequencies is known the other one can be determined.

(C) Frequency ratio 1:2 :

Let two rectangular SHMs of frequencies in the ratio 1:2 differing in phase by δ be represented by

$$x = a\cos\omega t \qquad \ldots(2.5\text{-}9)$$

$$y = b\cos(2\omega t + \delta) \qquad \ldots(2.5\text{-}10)$$

$$\therefore \quad \frac{y}{b} = \cos 2\omega t.\cos\delta - \sin 2\omega t.\sin\delta$$

$$= (2\cos^2\omega t - 1)\cos\delta - 2\sin\omega t.\cos\omega t.\sin\delta$$

Using Eq. (2.5-9) we get

$$\frac{y}{b}=\left(2\frac{x^2}{a^2}-1\right)\cos\delta-2\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\sin\delta$$

or,

$$\left[\frac{y}{b}+\cos\delta-\frac{2x^2}{a^2}\cos\delta\right]^2=\frac{4x^2}{a^2}\left(1-\frac{x^2}{a^2}\right)\sin^2\delta \qquad \ldots(2.5\text{-}11)$$

This is an equation of fourth degree in x and, in general, represents closed curve having two loops. For a given value of δ the exact nature

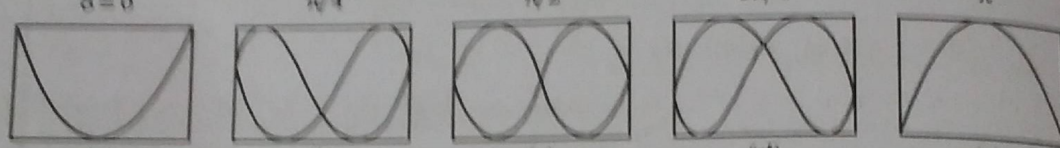

Fig. 2.5-2 : Resultant pattern of two rectangular SHMs having frequency ratio 1 : 2

of the curve can be traced. For example, if $\delta=0$ then Eq. (2.5-11) reduces to

$$\left(\frac{y}{b}+1-\frac{2x^2}{a^2}\right)^2=0$$

It represents two coincident parabolas [Fig. 2.5-2(a)] given by

$$x^2=\frac{a^2}{2b}(y+b) \qquad \ldots(2.5\text{-}12)$$

If $\delta=\pi/2$ Eq. (2.5-11) reduces to

$$\frac{4x^2}{a^2}\left(\frac{x^2}{a^2}-1\right)+\frac{y^2}{b^2}=0 \qquad \ldots \ldots(2.5\text{-}13)$$

This equation represents a curve containing two loops as shown in Fig. 2.5-2(c).

(d) Frequencies in any commensurate ratio :

If the frequency ratio of the rectangular SHMs becomes large then analytical method of obtaining the resultant pattern becomes very cumbersome. In that case one can use graphical method using the concept of rotating vectors.

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2.6 Lissajous figures :

The figures or curves formed by the superposition of two simple harmonic motions at right angles to each other are known as *Lissajous figures*. The shape of these curves depends on the ratio of frequencies as well as on the initial phase relationship of the component simple harmonic motions. The curves shown in Fig. 2.5-1 and 2.5-2 are examples of Lissajous figures.

Demonstration of Lissajous figures :

There are a number of experimental methods for obtaining Lissajous figures. These may be classified as mechanical, optical and electrical methods.

(i) Mechanical method :

A simple method to obtain Lissajous figures is to use Blackburn's pendulum. It consists of a heavy metal ring M with a sand-filled funnel G fitted on it. The metal ring is suspended by two long strings as shown in Fig. 2.6-1. A clip is used to catch the wires at C. The arrangement can be considered as a combination of two pendulums. For vibration in the plane of the strings it behaves like a pendulum of length l_1 and for vibration normal to the plane of the strings it behaves like a pendulum of length l_2 as indicated in Fig. 2.6-1. If the system is set into oscillation in an arbitrary way it will be under the action of two rectangular SHMs. The sand from the funnel trickles down on a paper placed below it and forms Lissajous figure.

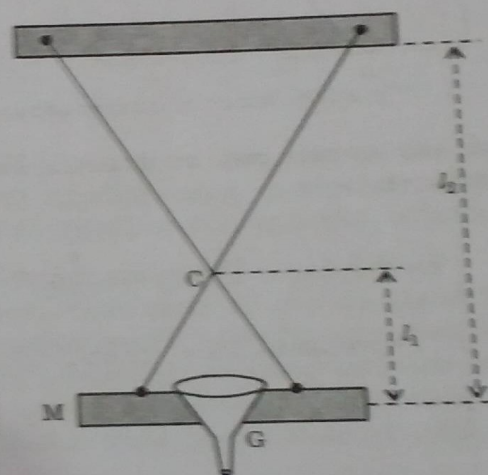

Fig. 2.6-1 : Blackburn's pendulum

(ii) Optical method :

In this method a strong beam of light is incident on a mirror M_1 attached to one prong of a tuning fork F_1 in such a way that after reflection from M_1 it falls on a second mirror M_2 attached to the prong of a second tuning form F_2. The tuning forks vibrate at right angles to one another. The final reflected light beam forms Lissajous figure on an opaque screen S.

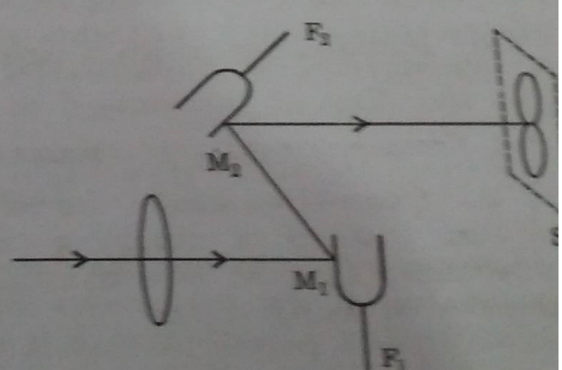

Fig. 2.6-2

(iii) Electrical method :

Most suitable method of demonstrating Lissajous figures is to use a modern cathode ray oscilloscope (CRO). Here a narrow beam of electrons from an electron gun is passed through two pairs of parallel metal plates arranged at right angles to each other. When electric field

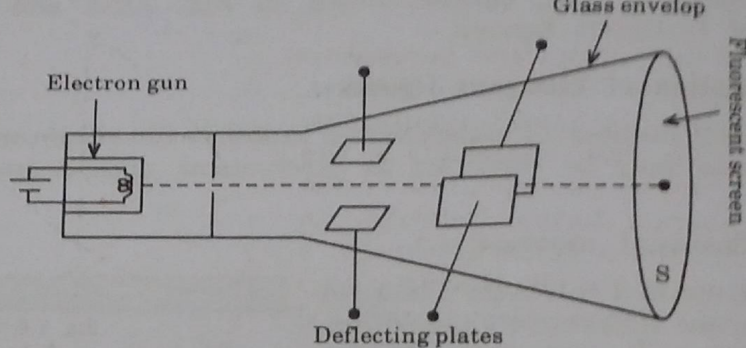

Fig. 2.6-3 : Cathod-ray oscilloscope

is set up one pair of plates deflects the electrons in horizontal direction and the other pair deflects them in vertical direction. The electrons finally impinge on a fluorescent screen and produce visible spot.

To display a Lissajous figure on the CRO screen two simple harmonic vibrations are at first converted into sinusoidal voltages with the help of two microphones. These voltages, after proper magnification, are applied to the two pairs of deflecting plates of the CRO. As the elcetrons move under the simultaneous action of two sinusoidal elcetric fields at right angles to each other, they traces out Lissajous figure on the CRO screen.

Use of Lissajous figures :

Lissajous figures have many important uses. For example, these figures may be used to find the ratio of two exactly commensurate frequencies of the component vibrations. For this purpose draw a horizontal and a vertical line to intersect the curve and find the number of intersections each line makes with curve. Now the ratio of the number of intersections of the horizontal line to that on the vertical line will give the ratio of the vertical to the horizontal frequency i.e.,

$$\frac{\upsilon_{\text{vertical}}}{\upsilon_{\text{horizonal}}} = \frac{\text{No. of cuts on horizontal line}}{\text{No. of cuts on vertical line}}$$

Lissajous figures can be used to compare to two nearly equal frequencies. If the frequencies differ slightly Lissajous figures change gradually and passes through a complete cycle of changes in a time T given by

$$T = \frac{1}{\Delta\upsilon}$$

So by measuring T we can find the difference $\Delta\upsilon$ of frequencies. Again, if one of the frequencies is known then the other can be found out.

Lissajous figures can also be used to measure the phase difference δ between two signals of same frequency. Depending on the value of δ the Lissajous figure will be an ellipse or one of its degenerate forms. Referring to Fig. 2.6-4 let the distance of the point of intersection of the ellipse with y-axis be A and maximum vertical displacement be B. Now from Eq. (2.5-3), $A = b\sin\delta$ and $B = b$. Hence

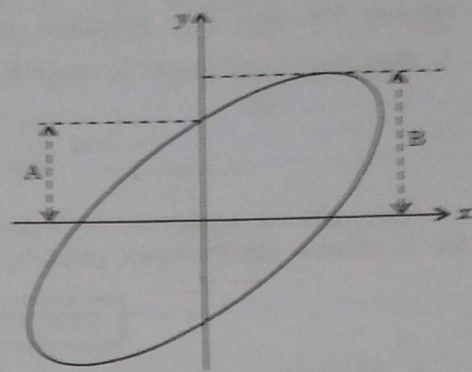

Fig. 2.6-4 : Measurement of phase differencec

$$\delta = \sin^{-1}(A/B)$$

Thus measuring A and B we can determine δ.

2.7 Superposition of a large number of SHMs :

There are some areas in physics where we require to find the superposition of a large number of SHMs. Let us consider one important situation in which we are to find the resultant of N SHMs having equal amplitude, equal frequency and equal successive phase difference.

Let us represent the successive SHMs by the complex exponential functions as

$$x_1 = ae^{j\omega t},\ \ x_2 = ae^{j(\omega t+\delta)},\ \ x_3 = ae^{j(\omega t+2\delta)},\ \ ...,\ \ x_N = ae^{j[\omega t+(N-1)\delta]}$$

The resultant complex vibration is given by

$$x = x_1 + x_2 + ...+x_N$$

$$= ae^{j\omega t}\left[1+e^{j\delta}+e^{2j\delta}+...+e^{j(N-1)\delta}\right]$$

$$= ae^{j\omega t}.\frac{1-e^{jN\delta}}{1-e^{j\delta}}$$

$$= ae^{j\omega t}.\frac{e^{\frac{jN\delta}{2}}}{e^{\frac{j\delta}{2}}}.\frac{e^{-\frac{jN\delta}{2}}-e^{\frac{jN\delta}{2}}}{e^{-\frac{j\delta}{2}}-e^{\frac{j\delta}{2}}}$$